RELATIONS AMONG WIND, TEMPERATURE, PRESSURE, AND DENSITY, WITH PARTICULAR REFERENCE TO MONTHLY AVERAGES

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ABSTRACT

The three-dimensional fields of horizontal wind, vertical wind, pressure, and density are expressed as functions of the temperature, the density, and the lapse rate at a given height. Computations with the formulas for monthly values are carried out and compared with observations. It is found, in agreement with other authors, that at a height of about 8 km. there exists a constant density surface. This allows us to express the horizontal wind as a function of the temperature and the lapse rate.

A kinematic method to compute the vertical wind is given. The numerical computations show that the use of a geostrophic wind is a good working method for the computations of vertical wind when one deals with average states over periods of a month.

1. INTRODUCTION

In previous papers [1], [2] a preliminary model to compute monthly mid-tropospheric and surface temperatures from the heat sources and sinks has been developed. The basic equations used are those of conservation of thermal energy in the vertically integrated troposphere and in the layer below the surface of the earth.

It is evident that the wind field is coupled with the temperature field and with the heat sources and sinks, and that it must, therefore, be generated within the model. Its obvious role in the conservation of thermal energy equation is in the advective terms, but it appears also in other important ways. For example, the heat lost from the surface of the oceans by evaporation and vertical turbulent transport, and the transport of energy by ocean currents depend on the surface wind.

The primary purpose of this paper is to develop formulas expressing the wind as a function of temperature in order to incorporate it as a variable in a thermodynamic model which is being developed as an extension of the previous work. However, the paper includes also other developments that may be of general interest to the reader.

2. HORIZONTAL WIND AS A FUNCTION OF TEMPERATURE AND DENSITY

We shall use the equations of hydrostatic equilibrium, perfect gas, and geostrophic wind, which are the following:

$$\frac{\partial p^*}{\partial z}=-\rho^* g \tag{1}$$

$$p^*=\rho^* RT^* \tag{2}$$

$$-fu^*=\frac{1}{\rho^*}\frac{\partial p^*}{\partial y} \tag{3}$$

$$fv^*=\frac{1}{\rho^*}\frac{\partial p^*}{\partial x} \tag{4}$$

where T^*, p^*, ρ^* are the three-dimensional fields of temperature, pressure, and density respectively; u^* and v^* are the components along the x and y axes respectively of the horizontal wind, the x axis points to the east, the y axis to the north, and the vertical z axis upward; g is the acceleration of gravity, R the gas constant, and f the Coriolis parameter.

We shall consider a layer in the troposphere in which the lapse rate will be assumed independent of height, but a function of the horizontal coordinates and time. Therefore

$$T^*=-\beta(z-H)+T \tag{5}$$

where β is the lapse rate, and T is the temperature at the arbitrarily chosen level H, which is also a function of the horizontal coordinates and time.

From (1), (2), and (5) it follows that

$$p^*=F_1 T^{*g/R\beta} \tag{6}$$

$$\rho^*=\frac{F_1}{R}T^{*(g/R\beta)-1} \tag{7}$$

where F_1 is an arbitrary function of x, y, and t. For $z=H$ the relation between pressure and temperature is obtained from (6) by dropping the star superscripts. Therefore

$$p^*=p(T^*/T)^{g/R\beta} \tag{8}$$

Similarly

$$\rho^*=\rho(T^*/T)^{(g/R\beta)-1} \tag{9}$$

where ρ is the density at $z=H$ and $p=\rho RT$ is the pressure also at $z=H$. Substituting (8) and (9) in (3) and (4), we obtain

$$u^*=-\frac{RT^*}{fT}\frac{\partial T}{\partial y}-\frac{RT^*}{f\rho}\frac{\partial \rho}{\partial y}-\frac{g}{f}\frac{\partial H}{\partial y}+\frac{(H-z)g}{fT}\frac{\partial T}{\partial y}$$
$$+\frac{g}{f}\left[z-H+\frac{T^*}{\beta}\ln\left(\frac{T^*}{T}\right)\right]\frac{1}{\beta}\frac{\partial \beta}{\partial y} \tag{10}$$

$$v^*=\frac{RT^*}{fT}\frac{\partial T}{\partial x}+\frac{RT^*}{f\rho}\frac{\partial \rho}{\partial x}+\frac{g}{f}\frac{\partial H}{\partial x}-\frac{(H-z)g}{fT}\frac{\partial T}{\partial x}$$
$$-\frac{g}{f}\left[z-H+\frac{T^*}{\beta}\ln\left(\frac{T^*}{T}\right)\right]\frac{1}{\beta}\frac{\partial \beta}{\partial x} \tag{11}$$

Formulas (5), (8), (9), (10), and (11) express the three-dimensional fields of temperature, pressure, density, and horizontal wind as functions of the temperature, the density, and the lapse rate at the arbitrary surface H.

CASE 1. WHEN H IS THE HEIGHT OF AN ISOBARIC SURFACE

Of special interest is the case in which H is the height of an isobaric surface. In this case, since p is constant, from (2) it follows that the sum of the two first terms in the right side of equations (10) and (11) is zero. Therefore, the wind at any level in an atmospheric layer can be computed from the height and the temperature of an isobaric surface, and the mean lapse rate in the layer.

Using observed values of 500-mb. heights and 500-mb. temperatures, we shall carry out computations with mean monthly averages for February 1964 and February 1962. To avoid the use of arbitrary normals we shall compute the difference of these two cases as a test of computations for departures from normal.

Figures 1A and 1B show the zonal and meridional wind components for February 1964 at $z=3$ km. (which corresponds roughly to the 700-mb. level), computed from formulas (10) and (11), with 500-mb. heights, 500-mb. temperatures, and a constant lapse rate equal to 6.5° C. km.$^{-1}$ These results are in good agreement with the corresponding wind components computed geostrophically directly from the 700-mb. heights, which are shown in figures 1C and 1D.

Figures 2A and 2B show sea level pressure values ($z=0$) computed with formula (8), from 500-mb. temperature and heights using the same constant lapse rate as above. Figure 2A shows the values for February 1964 and figure 2B the difference, February 1964 minus February 1962. Comparison with the corresponding observed values, given in figures 2C and 2D, shows good agreement.

CASE 2. WHEN THE DENSITY IS CONSTANT AT A GIVEN CONSTANT HEIGHT

In this case the second and third terms of the right side of equations (10) and (11) are equal to zero. Therefore, the horizontal wind at any level in an atmospheric layer can be computed from the temperature at the level $z=H$, where the density is assumed constant, and from the mean lapse rate in the layer.

It is easy to see that the solution will depend on the choice of the level H of assumed constant density. Computations using $H=5.5$ km. yield horizontal winds which are about 30 percent higher than the geostrophic winds at the corresponding levels computed from the heights of isobaric surfaces.

If we raise the level of assumed constant density the wind intensity in the solution increases, and at a level equal to about 8 km. we find the best agreement with the geostrophic wind.

Figures 3 and 4 show the results of such computations. Figures 3A and 3B are the values for February 1964 of the zonal and meridional wind components at 500 mb., computed using formulas (10) and (11) with constant ρ, $H=8$ km., and T equal to the temperature at 5.5 km. minus the lapse rate multiplied by 2.5 km. Figures 3C and 3D are the corresponding values computed geostrophically from 500-mb. heights.

Figures 4A and 4B show the values of the differences February 1964 minus February 1962 of the zonal and meridional wind components at 500 mb. computed assuming a constant density at 8 km.; and figures 4C and 4D the corresponding values computed geostrophically from 500-mb. heights.

This agreement suggests that at the 8-km. level the contribution of the terms containing $\partial\rho/\partial y$ and $\partial\rho/\partial x$ in formulas (10) and (11) is a minimum. Therefore, this level seems to be the best approximation to a constant density surface.

This result can also be obtained by a direct computation of the density from formula (9), by using 500-mb. temperatures and heights, and taking $\rho=RT/500$ mb. Computations with a constant lapse rate equal to 6.5°C. or with a mean observed lapse rate show that at the surface of the earth the air density increases from lower to higher latitudes. This gradient of the density decreases with height, and at a level of about 8 km. it is equal to zero. Above this level the density gradient as a function of latitude is reversed (the density then decreases from lower to higher latitudes) and increases with height.

Searching in the extensive meteorological literature one finds that several authors ([3], [4], [5], [8], [9]) have found already from observations that indeed there exists a constant density surface (isopycnic surface) at a height of about 8 km.

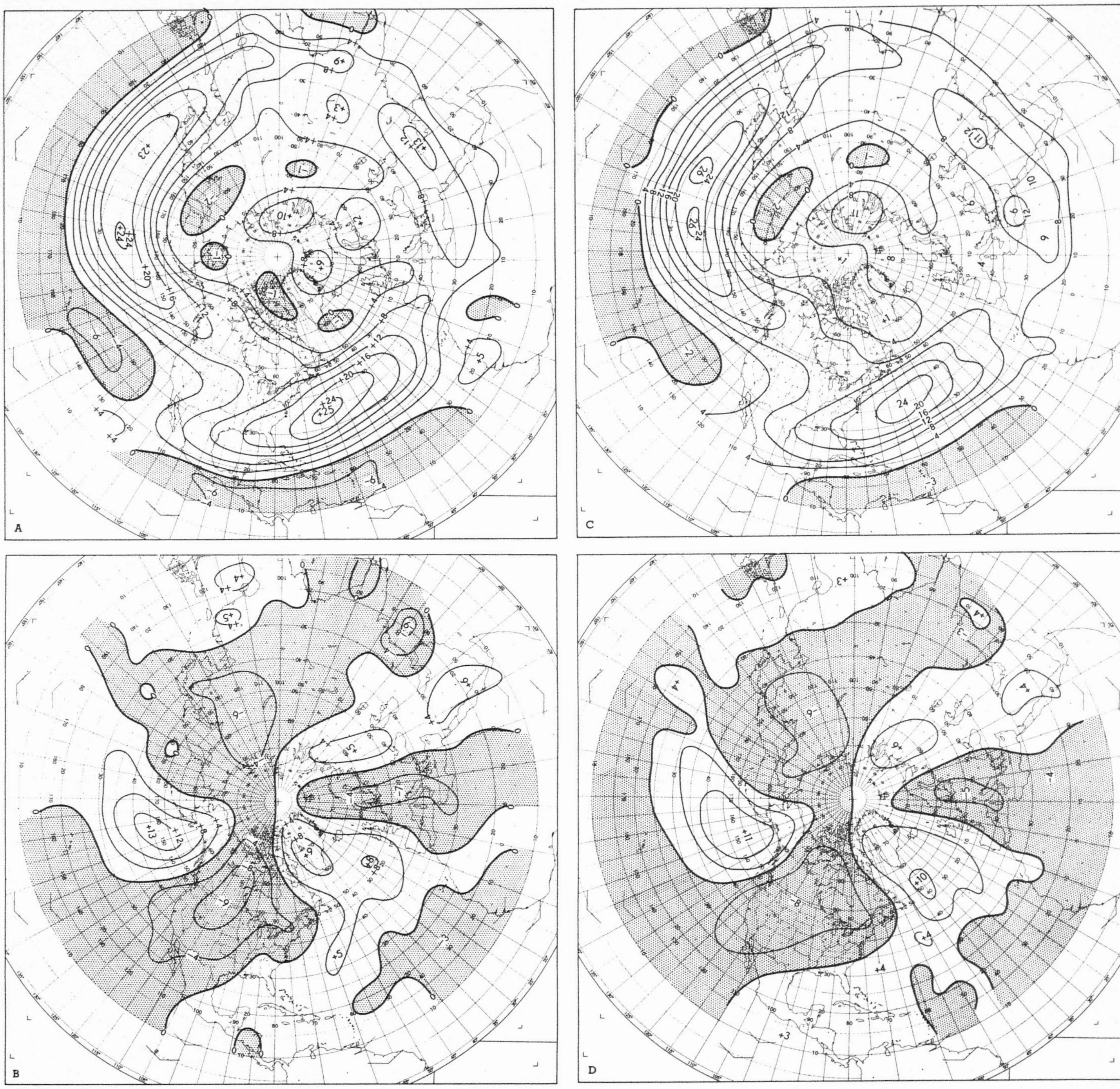

FIGURE 1.—The horizontal wind for February 1964 at 700 mb., in meters per second. (A) and (B) are the values of the zonal and meridional components respectively, computed with the formulas from 500-mb. values. (C) and (D) are the corresponding values computed geostrophically from 700-mb. heights.

3. COMPUTATION OF THE VERTICAL WIND

The problem of computing the vertical wind from observed meteorological data has been the subject of several papers. Panofsky [7] has given an excellent survey article and more recently Jensen [6] made computations for the whole Northern Hemisphere using 1958 data. For a complete bibliography on this subject the reader is referred to the references in both of these publications.

The continuity equation contains the vertical wind and its first derivative with respect to the vertical coordinate.

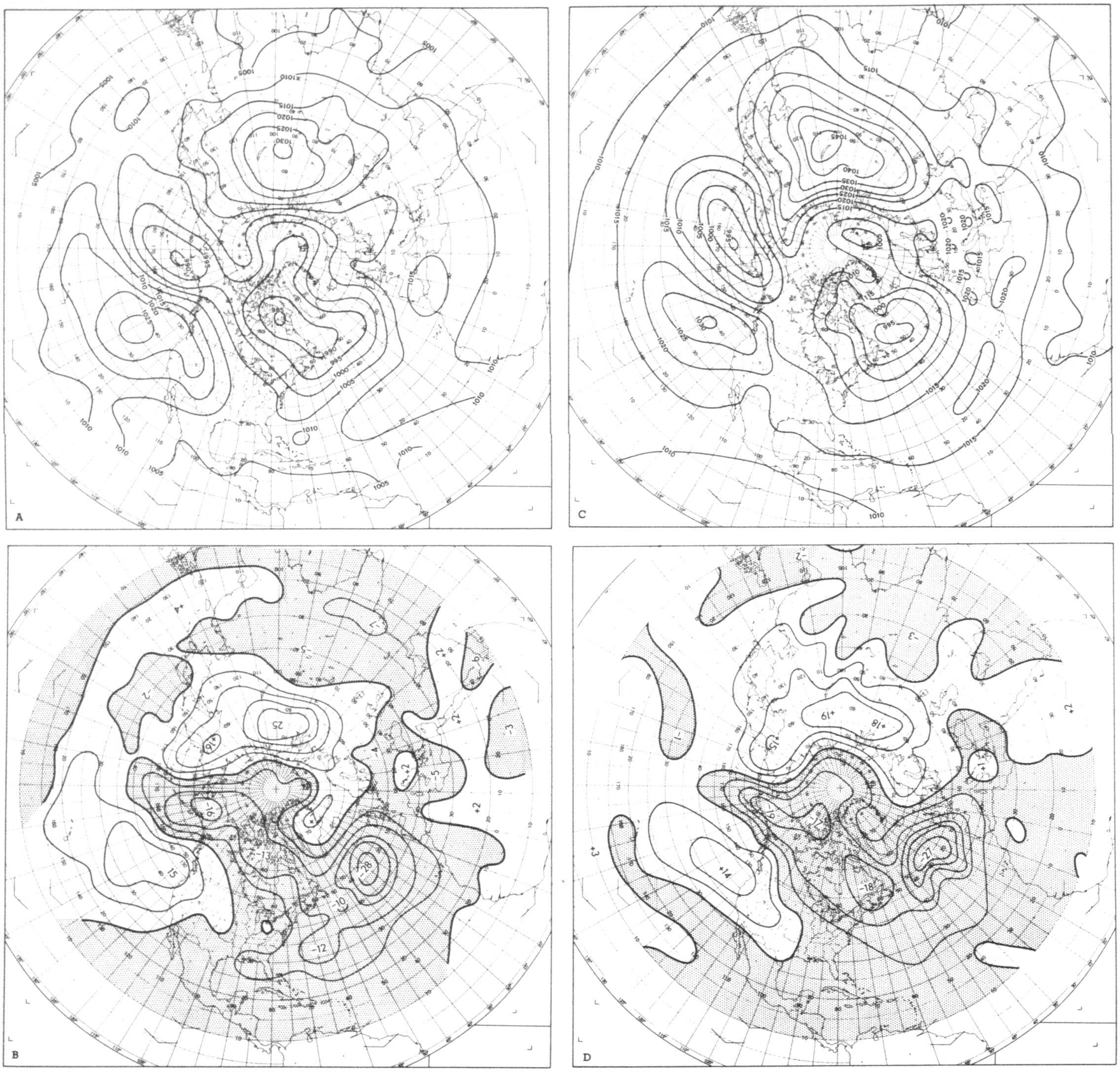

FIGURE 2.—Sea level pressure, in mb. (A) shows the values for February 1964 and (B) the values of the difference, February 1964 minus February 1962, computed from 500-mb. temperatures and heights and a constant lapse rate. (C) and (D) are the corresponding observed values.

Therefore by solving a linear first order differential equation it is possible to obtain a formula for the vertical wind. The solution depends on the horizontal component of the wind and the density and its local rate of change.

The continuity equation can be written as

$$\frac{\partial w^*}{\partial z}+Q_1 w^*=Q_2 \tag{12}$$

where

$$Q_1=\frac{1}{\rho^*}\frac{\partial \rho^*}{\partial z} \tag{13}$$

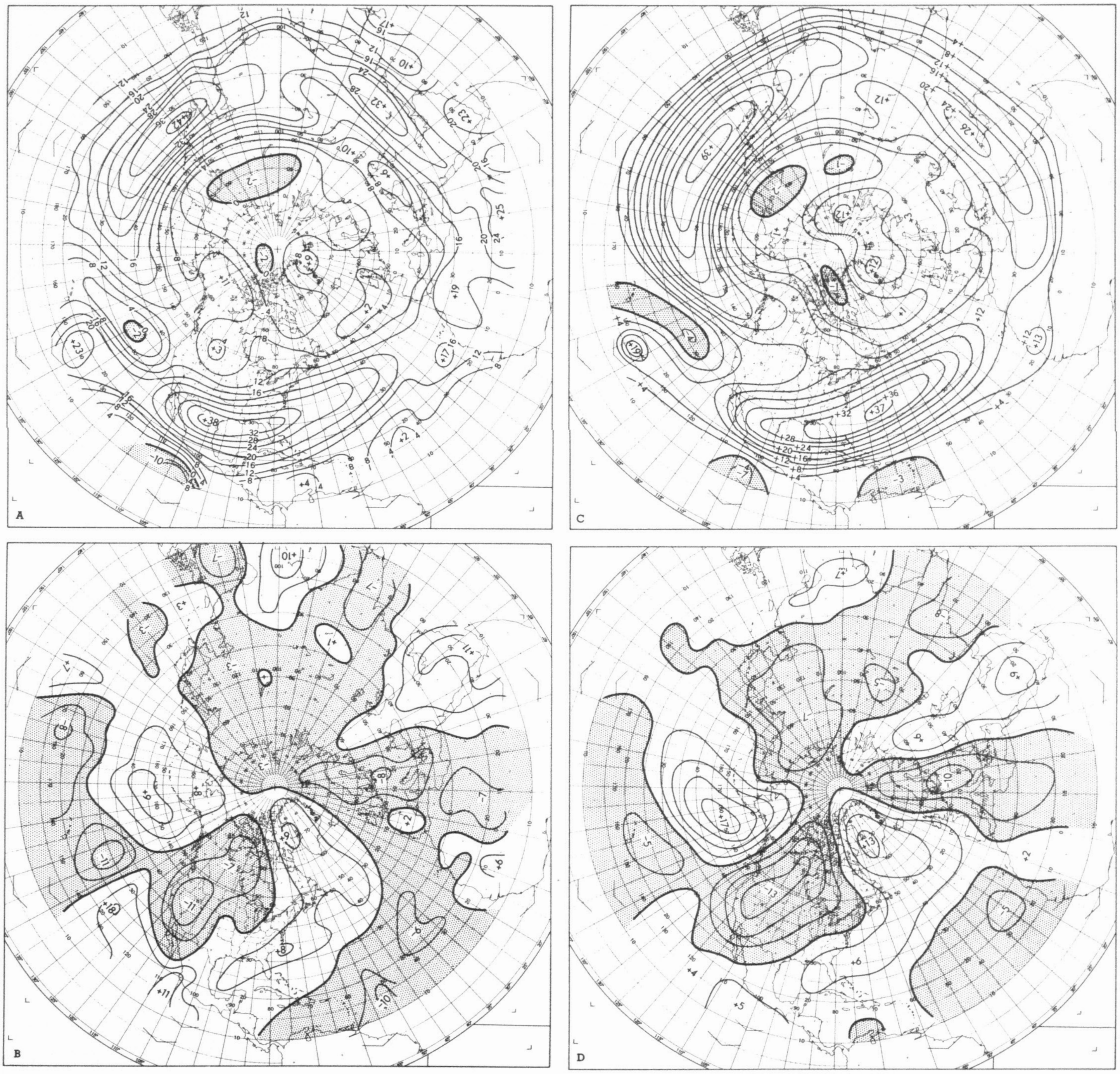

FIGURE 3.—Horizontal wind at the 500-mb. height for February 1964 in meters per second. (A) and (B) are the values of the zonal and meridional wind components respectively, computed from the temperature at a height of 5.5 km. assuming a constant density at a height of 8 km. (C) and (D) are the corresponding values computed geostrophically from 500-mb. heights.

$$Q_2=-\frac{1}{\rho^*}\left(u^*\frac{\partial\rho^*}{\partial x}+v^*\frac{\partial\rho^*}{\partial y}\right)-\left(\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}\right)-\frac{1}{\rho^*}\frac{\partial\rho^*}{\partial t} \quad (14)$$

ρ^* is the density, u^* and v^* are the components along the x and y axes, respectively, of the horizontal wind, the x axis points to the east and the y axis to the north. w^* is the vertical wind, z is the vertical coordinate measured from the sea level, and t is the time variable.

If we know u^*, v^*, ρ^*, and $\partial\rho^*/\partial t$ then we can compute w^* from (12). The solution is

$$w^*=\frac{\int Q_2Q_3dz}{Q_3}+\frac{C_2}{\rho^*} \quad (15)$$

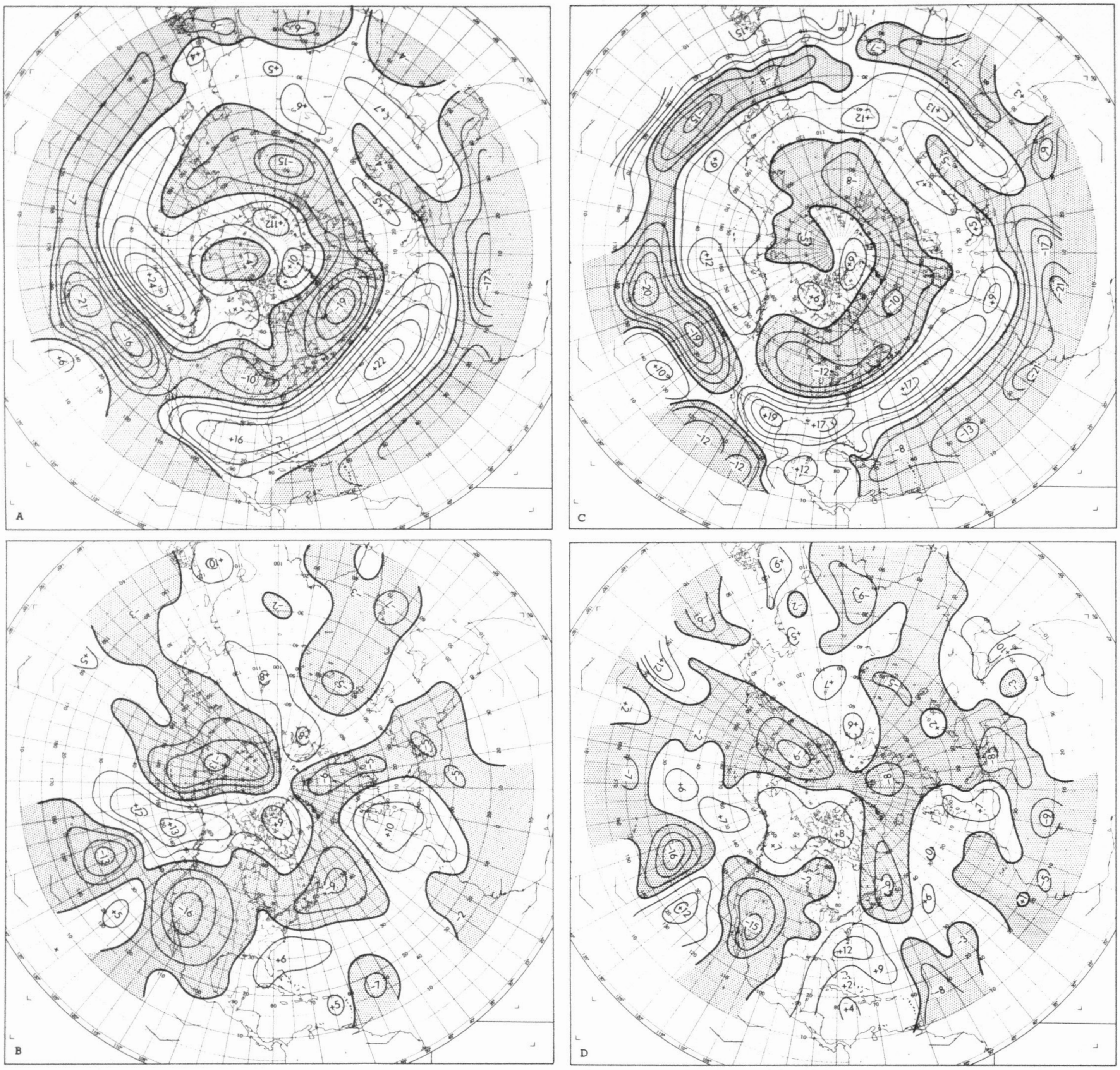

FIGURE 4.—Horizontal wind differences, February 1964 minus February 1962, at 500 mb., in meters per second. (A) and (B) are the values of the differences of the zonal and meridional wind components, respectively, computed from a temperature at a height of 5.5 km., assuming a constant density at a height of 8 km. (C) and (D) are the corresponding values computed geostrophically from 500-mb. heights.

where C_2 is an arbitrary function of the horizontal coordinates and time and

$$Q_3=e^{\int Q_1 dz} \tag{16}$$

From the condition that the vertical wind is known at the lower boundary of the considered atmospheric layer it follows that

$$C_2=(\rho^*)_{z=H_L}\left[w_L-\left(\frac{\int Q_2Q_3dz}{Q_3}\right)_{z=H_L}\right] \tag{17}$$

FIGURE 5.—Vertical wind at a height of 7 km. above sea level. (A) shows the computed values for February 1964, using the geostrophic divergence kinematical method and (B) the values for January 1958 computed by Jensen [6] with the adiabatic method. (C) is the component due to the horizontal variations of density; and (D) is the component that depends on $J\ (f, \beta)$.

where H_L is the height of the lower boundary and w_L is the vertical wind at $z=H_L$.

Substituting (17) in (15) we obtain

$$w^*=F+\frac{(\rho^*)_{z=H_L}}{\rho^*}[w_L-(F)_{z=H_L}] \tag{18}$$

where $F=(\int Q_2Q_3dz)/Q_3$ and $\alpha=g/R\beta$.

Assuming $\partial\rho^*/\partial t=0$ and using (5), (9), (10), and (11), we obtain from (18), after integration, the following formula

$$w^*=\left(\frac{T_L}{T^*}\right)^{\alpha-1}w_L+\left[G_1-\left(\frac{T_L}{T^*}\right)^{\alpha-1}(G_1)_{z=H_L}\right]J(f,H)$$
$$+\left[G_2-\left(\frac{T_L}{T^*}\right)^{\alpha-1}(G_2)_{z=H_L}\right]J(f,T)$$
$$+\left[G_3-\left(\frac{T_L}{T^*}\right)^{\alpha-1}(G_3)_{z=H_L}\right]J(f,\beta)$$
$$+\left[G_4-\left(\frac{T_L}{T^*}\right)^{\alpha-1}(G_4)_{z=H_L}\right]J(\beta,H)$$
$$+\left[G_5-\left(\frac{T_L}{T^*}\right)^{\alpha-1}(G_5)_{z=H_L}\right]J(\beta,T) \quad (19)$$

where $T_L=(T^*)_{z=H_L}$ and

$$G_1=RT^*/f^2$$

$$G_2=RT^*F_2/f^2T$$

$$G_3=\frac{RT^*}{f^2\beta}\left\{\frac{gT^*}{\beta(g+R\beta)}\left[\frac{1}{\alpha+1}-\ln\left(\frac{T^*}{T}\right)\right]-F_2\right\}$$

$$G_4=\frac{RT^*}{f\beta}\left[1-\alpha\ln\left(\frac{T^*}{T}\right)\right]$$

$$G_5=\frac{RT^*}{\beta fT}\left\{(1-\alpha)\left[F_2\ln\left(\frac{T^*}{T}\right)-\frac{z-H}{\alpha}-\frac{RT^*(2\alpha+1)}{g(\alpha+1)^2}\right]\right.$$
$$-\frac{T}{\beta}+\frac{T^*\alpha}{\beta(\alpha+1)}+\frac{T}{\beta\alpha}-\frac{T}{\beta}\ln\left(\frac{T^*}{T}\right)$$
$$\left.+F_2-\frac{gT^*}{\beta(g+R\beta)}\left[\frac{1}{\alpha+1}-\ln\left(\frac{T^*}{T}\right)\right]\right\}$$

where $F_2=z-H+T^*/\beta(\alpha+1)$.

The functions $J(f, H)$, $J(f, T)$, $J(f, \beta)$, $J(\beta, H)$ and $J(\beta, T)$ are Jacobians (e.g., $J(\beta, T)=(\partial\beta/\partial x)\ (\partial T/\partial y)-(\partial\beta/\partial y)\ (\partial T/\partial x)$).

We shall carry out computations with a one-layer model atmosphere. Therefore in this case the lower boundary of the layer is the surface of the earth.

In formula (11) the term $(T_L/T^*)\alpha^{-1}w_L$, which depends on the surface wind, is negligible, except in mountainous terrain, where it can become very important.

The terms containing $J(f, H)$, $J(f, T)$, and $J(f, \beta)$, which depend on the variation of the Coriolis parameter, correspond to the solution when $\partial\rho^*/\partial x$ and $\partial\rho^*/\partial y$ are neglected, as is $\partial\rho^*/\partial t$.

The terms containing $J(\beta, H)$ and $J(\beta, T)$ which depend on the horizontal variations of the lapse rate ($\partial\beta/\partial x$ and $\partial\beta/\partial y$), are due to the horizontal variations of the density ($\partial\rho^*/\partial x$ and $\partial\rho^*/\partial y$).

To estimate the total vertical wind and its components we need to know the temperature at the isobaric surface H, the generalized lapse rate in the layer, β, and the vertical wind at the surface, w_L.

We will choose H as the 500-mb. height. The vertical wind at the surface of the earth will be taken as zero. Therefore, in mountainous terrain we expect erroneous computed values. However, on a hemispheric scale the possible errors introduced with this simplification are geographically fixed and of relatively small extent.

As generalized lapse rate, β, in this one-layer model, we will use the lapse rate from 700 to 300 mb.

Figure 5A is the total vertical wind computed using formula (19) for February 1964 at a height of 7 km.

Figure 5B is the total vertical wind in the layer 500–300 mb., which corresponds roughly to the same height of 7 km., for January 1958 computed by Jensen [6], using an adiabatic method, which is completely independent of our kinematic method. Figure 5B shows similar general patterns and values as figure 5A. Although we are comparing our results with a different winter month of a different year, this rough agreement verifies the consistency and soundness of our approach.

Figure 5C shows the contribution to the solution by the variations of the density. Its comparison with figure 5A shows that the most important component of the vertical wind is the one due to the horizontal divergence ($\partial u^*/'\partial x+\partial v^*/\partial y$).

Figure 5D shows the term of the solution that contains $J(f, \beta)$, which is the contribution of the horizontal divergence component from the horizontal variations of the lapse rate. It is in general negligibly small compared with figure 5A, except in lower latitudes where, because of the variation of the Coriolis parameter, it can become important.

In the above computations, for the sake of simplicity we considered a one-layer model. However it is evident that formula (19) can also be applied successively to each of the individual layers of a multiple-layer model to obtain the vertical motion in each layer. To do this it is necessary to specify the vertical wind and the temperature at the surface of the earth and the temperature and height at the isobaric surfaces which separate the layers.

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